Linear algebra summaries

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# Determinant

Unlike matrix which is a table of numbers, a determinant is just one number:

Note a determinant always has the same number of columns and rows, in other words, it is a square shape.

## Cofactor expansion

Let be the minor of the determinant after deleting the i-th row and j-th column:

The associated cofactor is then defined by . It is found that can be expanded according to a certain column or row. For instance, the expansion according to the 1st column is the following:

## Triangle determinant (determinant of a triangle matrix)

If is the determinant of a triangle matrix, then:

## Properties

### A determinant and its transpose have the same value:

### A determinant multiplied by a number is equivalent to the scenario in which the elements in a certain row or column are all multiplied by , i.e.,

Note this property is different from matrix.

### The value of the determinant changes sign when two columns or rows are swapped

### If all elements in one row (column) are times that of the other, than the value of the determinant is 0

### If all elements in one row (column) can be broken to the sum to two other numbers, , then:

### Multiplying one row (column) by and adding it to another row (column) does not change the value of the determinant

## [Cramer’s rule](https://en.wikipedia.org/wiki/Cramer%27s_rule)

One example, let:

Then the solutions are:

Cramer’s rule is quite computationally expensive and hence is not adopted when solving the linear equations numerically.

## [Relation to the area of a parallelogram or the volume of a parallelepiped](https://mathinsight.org/relationship_determinants_area_volume)

### Relation to the area of a parallelogram

The area of a parallelogram spanned by the vectors and is the magnitude of :

Assume bothand lie in the same plane so that , hence:

Therefore, the area of the parallelogram is given:

### Relation to volume of a parallelepiped

The volume of a parallelepiped spanned by the vectors , , and is the magnitude of :

Hence, the volume of a parallelepiped can be expressed as:

# Matrix

Unlike the determinant which is essentially a number, the matrix is a table of numbers. The identity matrix is equivalent to 1 in numbers.

## Properties

### Addition and multiplication by numbers

Matrix addition and number multiplication have the following properties:

Note multiplying a matrix by a number is equivalent to multiplying each element of this matrix by this number. This is different from multiplying a determinant by a number.

### Matrix multiplication

Let , then each element in is the inner product of the row vectors in and column vectors in :

Here, and are the row and column vectors of and , respectively. Matrix multiplication has the following properties:

In addition, it’s possible that , even though and .

### Matrix transpose

Matrix transpose has the following properties:

Given that , the last property is exceptionally interesting. It can be generalized to finite number of matrices:

Both and are symmetric matrices.

Any square matrix can be decomposed to the sum of a symmetric and an anti-symmetric matrix:

### Matrix inverse

Definition: if , then is invertible and the inverse of is.

For to be invertible, the sufficient and necessary condition is . Under such a circumstance:

Here is’sadjugate matrix. If , then has the following expression:

Note is the transpose of the matrix formed by the cofactors of each element in .

Matrix inverse has the following properties:

Note the last one is very similar to matrix transpose. It can be quickly proved in the following:

Similar to matrix transpose, this property can be generalized to finite number of matrices:

### Block matrix

Block matrix inverse: let , then .

Block matrix multiplication: in addition, let , then . Here . In other words, the rule is the same as ordinary matrix multiplication.

Some special cases: if, , then:

### Elementary matrix operations

There are three elementary matrix operations, corresponding to three elementary matrices:

* Interchange two rows (or columns)
* Multiply each element in a row (or column) by a non-zero number
* Multiply a row (or column) by a non-zero number and add the result to another row (or column)

Row-wise (column-wise) elementary operations on a matrix are equivalent to pre-multiply (post-multiply) by the corresponding elementary matrices.

Through elementary matrix operations, any matrix can be converted to the standard form:

Here, and are multiplication of a series of elementary matrices that correspond to row-wise and column-wise elementary operations, respectively. It’s not hard to see that both and are invertible.

A special case is when is invertible, , then , which means the invertible matrix is essentially the multiplication of finite number of elementary matrices. This introduces a convenient approach to calculate the inverse matrix. If is invertible, then its inverse can be expressed by :

This indicates the same row-wise operations that convert to an identity matrix will convert the identity matrix to its inverse!

See the following example:

Hence, the inverse of is:

### Matrix rank

Properties of matrix’s rank:

* If is n×n square matrix,

Note for any matrix, its column-wise rank equals to its row-wise rank.

# Vector space

## Maximal linearly independent subset

The number of vectors in the maximal linearly independent subset of a vector set is this set’s rank. If two vector sets are equivalent, then they share the same rank.

Let , . If any can be linearly expressed using , and , then must be linearly dependent.